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A COMPUTATIONAL ALGORITHM FOR THE EIGENVECTORS OF A SINGULAR MATRIX

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In this paper a computational, non-iterative type algorithm is given to obtain the eigenvectors of a singular matrix. The algorithm yields a (smaller) nonsingular matrix that has exactly the same nonzero eigenvalues as the given singular matrix and only these eigenvalues. The eigenvectors for the singular matrix are constructed from the eigenvectors of the nonsingular matrix. The eigenvectors associated with the zero eigenvalues of the singular matrix are an immediate consequence of the algorithm.

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I. INTRODUCTION

In this paper the eigenvalues and the eigenvectors of a singular matrix, A, are investigated.[1] Present techniques for determining the eigenvalues of a matrix require that the matrix be nonsingular or that some iterative procedure such as root searching techniques be applied to the characteristic equation. Here, a computational (non-iterative type) algorithm is given to construct a (smaller) nonsingular matrix, Q, that has exactly the same nonzero eigenvalues as A and only these eigenvalues. Thus, a standard technique such as the QR algorithm[2] can be applied to the matrix Q to evaluate the eigenvalues and eigenvectors of Q.

The algorithm given in this paper yields the eigenvectors of the nonzero eigenvalues of A, which are obtained from the eigenvectors of Q, and the rank of the matrix A. The eigenvectors associated with the zero eigenvalues of A are an immediate consequence of the algorithm.

II. MOTIVATION OF THE ALGORITHM

Let $A = (a_{ij})$ be an $n \times n$ matrix and U an $n \times n$ nonsingular matrix. Define $W = U^{-1}AU$. Let λ and X be an eigenvalue and an eigenvector, respectively, of A, then

$$\begin{aligned} AX &= \lambda X \\ U^{-1}A(UU^{-1})X &= \lambda U^{-1}X \\ W(U^{-1}X) &= \lambda(U^{-1}X), \end{aligned} \tag{1}$$

that is, λ and $U^{-1}X$ are an eigenvalue and eigenvector of W. Conversely, if μ and Z are an eigenvalue and an eigenvector of W, then μ and UZ are an eigenvalue and an eigenvector of A.

Suppose A is of rank r and there exists a U such that W can be partitioned into the following form:

[1] The algorithm given in this paper was developed by the author in response to a problem posed by H. McCoy, TRASANA, to obtain the eigenvalues and eigenvectors of a special singular matrix.

[2] Wilkinson, J. H., The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.

$$W = U^{-1}AU = \begin{pmatrix} Q_r & O_1 \\ B & O_2 \end{pmatrix}, \tag{2}$$

where Q_r is an $r\times r$ matrix, O_1 and O_2 are $r\times(n-r)$ and $(n-r)\times(n-r)$ zero matrices, respectively, and B is an $(n-r)\times r$ matrix. Let $\lambda \neq 0$ be an eigenvalue of W, and of A, and Z be the associated eigenvector. If the vector Z, as a matrix, is partitioned into Z_r, a $r\times 1$ matrix, and Y, a $(n-r)\times 1$ matrix, then we can write the following matrix equation

$$\begin{pmatrix} Q_r & O_1 \\ B & O_2 \end{pmatrix}\begin{pmatrix} Z_r \\ Y \end{pmatrix} = \lambda\begin{pmatrix} Z_r \\ Y \end{pmatrix} \tag{3}$$

as

$$Q_r Z_r = \lambda Z_r \tag{4}$$

and

$$BZ_r = \lambda Y, \tag{5}$$

that is, λ and Z_r are an eigenvalue and eigenvector of the matrix Q_r. Thus, the nonzero eigenvalues of W, therefore for A, are the eigenvalues (nonzero) of Q_r. The associated eigenvectors of A are given by

$$X = UZ = U\begin{pmatrix} Z_r \\ Y \end{pmatrix} = U\begin{pmatrix} Z_r \\ \lambda^{-1}BZ_r \end{pmatrix}, \tag{6}$$

which are determined from the eigenvectors of Q_r.

Conversely, if $\lambda \neq 0$ is an eigenvalue of Q_r and Z_r the eigenvector, then it follows that X, which is defined by (6), is an eigenvector of A and λ is the eigenvalue.

The eigenvectors of A associated with the zero eigenvalues are immediate. Define the column vectors

$$E_i = \begin{pmatrix} \delta_{1,r+i} \\ \delta_{2,r+i} \\ \vdots \\ \delta_{n,r+i} \end{pmatrix}, \quad i = 1, 2, \ldots, n-r, \tag{7}$$

where δ_{kl} is the Kronecker delta function. Then $WE_i = 0$, that is, E_i, $i = i, 2, \ldots, n-r$, are the associated eigenvectors for the zero eigenvalues of W. Thus, UE_i, $i = 1, 2, \ldots, n-r$, are the eigenvectors of the zero eigenvalues of A; but these eigenvectors are just the column vectors represented by the last n-r columns of the matrix U.

Since the rank of A is r, it follows that $\{UE_i\}$ is the total set of (distinct) eigenvectors associated with the zero eigenvalue. If Q_r is singular, this implies that the characteristic equation has a zero of multiplicity greater than n-r; but no greater collection of eigenvectors. The process would then be applied to Q_r to determine a matrix of smaller order that has the same nonzero eigenvalues as A.

III. EXISTENCE OF THE MATRIX U

We will now show the existence of the matrix U. Let A_i, $i = 1, 2, \ldots, n$, denote the column vectors represented by the columns of the matrix A. The definition and notation for the inner product of any two vectors is given by

$$(A_i, A_j) = \sum_{k=1}^{n} a_{ki}\bar{a}_{kj} , \tag{8}$$

where $\bar{a}$ denotes the complex conjugate of a.

We will now apply the Gram-Schmidt orthogonalization process (see [3]) to the set of vectors $\{A_i\}$. If A_1 is the zero vector, then interchange A_1 and A_n. Let U_1 be the elementary matrix that interchanges column one and column n, when post matrix multiplication is applied (this is the identity matrix with the first and nth columns interchanged). If, after this interchange, A_1 is zero, then interchange A_1 and A_{n-1}, where U_2 is the appropriate elementary matrix. Continue until A_1 is not the zero vector. Define $V_1 = A_1$ and

[3] Berberian, S. K., Introduction to Hilbert Space, Oxford University Press, New York, 1961.

$$V_2 = A_2 - \alpha_{21}V_1 , \tag{9}$$

where $\alpha_{21} = \dfrac{(A_2,V_1)}{(V_1,V_1)}$.

If V_2 is the zero vector, interchange A_2 and A_k, where A_{k+1} was the last vector interchanged; and U_j the appropriate elementary matrix. Define U_{j+1} to be the elementary matrix that adds $-\alpha_{21}$ times the second column to column k. This is given by appending to the identity matrix $-\alpha_{21}$ in row 2 and column k. Construct a new V_2. If V_2 is not the zero vector, then continue in the construction until a zero vector is generated, that is, define

$$V_i = A_i - \sum_{j=1}^{i-1}\alpha_{ij}V_j , \tag{10}$$

where $\alpha_{ij} = \dfrac{(A_i,V_j)}{(V_j,V_j)}$, $j = 1, \ldots, i-1$. If V_i is the zero vector, interchange A_i with the last vector not interchanged, say A_m, and assign an appropriate elementary matrix U_k. Since V_j, $j = 1, \ldots, i-1$, can be written in terms of $A_1, \ldots, A_j$, then A_i can be written in the following form:

$$A_i = \beta_{i1}A_1 + \ldots + \beta_{i,i-1}A_{i-1} . \tag{11}$$

Furthermore, let U_{k+j}, $j = 1, \ldots, i-1$, be the elementary matrices which adds $-\beta_{ij}$ times column j to column m. Continue this construction for V_i until A_{i+1} was the last vector interchanged.

Define U to be the product of the elementary matrices generated above, that is,

$$U = U_1U_2 \ldots U_t . \tag{12}$$

Then AU has the form:

$$AU = \begin{pmatrix} Q & 0_1 \\ R & 0_2 \end{pmatrix} . \tag{13}$$

For example, if V_i is the zero vector, then

$$A_i = \sum_{j=1}^{i-1} \beta_{ij} A_j \quad . \tag{14}$$

The product of the associated elementary matrices, say $U_k, \ldots, U_{k+i-1}$, would interchange column i with column m and substract β_{ij} times column j, j = 1, . . ., i-1, from column m. This would result in a zero column in column m.

If U_p interchanges column i with column j, then premultipliction by U_p^{-1} would interchange row i with row j; and if U_q would add $-\beta$ times column i to column j, U_q^{-1} would add β times row j to row i. Thus,

$$U^{-1} = U_t^{-1} \ldots U_2^{-1} U_1^{-1} \tag{15}$$

would result in row operations. Therfore, $U^{-1}AU$ would have the same form as (13), that is,

$$U^{-1}AU = \begin{pmatrix} Q_r & O_1 \\ B & O_2 \end{pmatrix} \quad . \tag{16}$$

The resulting collection of nonzero vectors, $A_1, \ldots, A_r$, under the Gram-Schmidt process, is the largest collection of independent vectors represented by the columns of A; therefore, r is the rank of A.

IV. CONSTRUCTION OF THE MATRIX U

In this section we will develop a computationally feasible algorithm for the construction of the matrices U, Q_r and B. It should be noted that the actual construction of U^{-1} will not be required to obtain the final result.

To start the construction, set $U = (u_{ij}) = (\delta_{ij})$, the identity matrix. If there exists a vector V_i equal to the zero vector, then from (14)

$$A_i = \sum_{j=1}^{i-1} \beta_{ij} A_j \ . \tag{18}$$

Assume A_{m+1} was the last column interchanged. If $u_{ii} = 1$, set $u_{ii} = 0$, $u_{im} = 1$, $u_{mi} = 1$ and $u_{mm} = 0$. If $u_{ii} = 0$, set $u_{m+1,i} = 0$, $u_{m+1,m} = 1$, $u_{mi} = 1$ and $u_{mm} = 0$. This interchanges column i and column m. Simultaneously, replace column i with column m in matrix A. Note that column m need not be replaced by column i, for this column is assumed to be the zero vector. In order to accomplish this, add $-\beta_{ij}$, $j = 1, \ldots, i-1$, to column m in the U matrix, where the ith row is determined from the one and only one nonzero element in the jth column of the U matrix. This nonzero element is unity. This construction is continued for i until i+1 was the last column interchanged. Thus, A has been reduced to the form of (13).

In order to obtain Q_r and B, similar operations must be done on rows of the reduced matrix A. An accounting must be kept on the column operations, for the row operations must be done in the same order, that is, if A has been postmultiplied by $U_1 U_2 \ldots U_t$, then A must be premultiplied by $U_t^{-1} \ldots U_2^{-1} U_1^{-1}$. Note that the row operations only involve the first r columns, for the remaining n-r columns are assumed to be zero vectors. Therefore, if a column operation involved interchanging column ℓ with column m, then the row operation would interchange row ℓ with row m. Similarly, if a column operation adds $-\beta$ times column ℓ to column m, then the row operation would add β times row m to row ℓ.

In order to construct β_{ij}, the vectors, V_i, must be generated, from which β_{ij} can be obtained recursively. From (10)

$$V_i = A_i - \sum_{k=1}^{i-1} \alpha_{ik} V_k \ , \tag{19}$$

where $\alpha_{ik} = \dfrac{(A_i, V_k)}{(V_k, V_k)}$. Suppose

$$V_k = A_k - \sum_{j=1}^{k-1} \beta_{kj} A_j \ , \quad k = 2, \ldots, i-1, \tag{20}$$

where $V_1 = A_1$. Then from (19) and (20)

$$V_i = A_i - \alpha_{i1}A_1 - \sum_{k=2}^{i-1}\alpha_{ik}(A_k - \sum_{j=1}^{k-1}\beta_{kj}A_j)$$

$$= A_i - \alpha_{i1}A_1 - \sum_{k=2}^{i-1}\alpha_{ik}A_k + \sum_{k=2}^{i-1}\sum_{j=1}^{k-1}\alpha_{ik}\beta_{kj}A_j$$

$$= A_i - \sum_{j=1}^{i-1}\alpha_{ij}A_j + \sum_{j=1}^{i-2}(\sum_{k=j+1}^{i-1}\alpha_{ik}\beta_{kj})A_j$$

$$= A_i - \alpha_{i,i-1}A_{i-1} - \sum_{j=1}^{i-2}(\alpha_{ij} - \sum_{k=j+1}^{i-1}\alpha_{ik}\beta_{kj})A_j \ . \tag{21}$$

Therefore,

$$\beta_{ij} = \begin{cases} \alpha_{ij} - \sum_{k=j+1}^{i-2}\alpha_{ik}\beta_{kj} & , \ j = 1, \ . \ . \ ., \ i-2 \\ \\ \alpha_{ij} & , \ j = i-1 \end{cases} \ . \tag{22}$$

V. SOME ILLUSTRATIVE EXAMPLES

A. Example 1.

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Then

$$V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} ,$$

which is not the zero vector.

$$V_2 = A_2 - \alpha_{21} V_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\alpha_{21} = \frac{(A_2, V_1)}{(V_1, V_1)} = \frac{2 + 1 + 6 + 0}{1 + 1 + 4 + 0} = \frac{3}{2}$$

and $\beta_{21} = \alpha_{21}$

$$V_3 = A_3 - \alpha_{31} V_1 - \alpha_{32} V_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - (-1) \begin{bmatrix} .5 \\ -.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\alpha_{31} = \frac{(A_3, V_1)}{(V_1, V_1)} = \frac{0 + 1 + 2 + 0}{6} = \frac{1}{2}$$

$$\alpha_{32} = \frac{(A_3, V_2)}{(V_2, V_2)} = \frac{0 + 0 + (-.5) + 0}{.25 + .25 + 0 + 0} = -1$$

and

$$\beta_{31} = \alpha_{31} - \alpha_{32}\beta_{21} = 2$$

$$\beta_{32} = \alpha_{32} \quad = -1.$$

Now V_3 is the zero vector. Therefore, replace column 3 with column 4 in A. We will assume column 4 is the zero column. For in theory, $-\beta_{31}$ times column 1 and $-\beta_{32}$ times column 2 are added to column 4. Hence, the reduced A matrix is

$$AU = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If the same column operations are applied to U, which was initally set to the identity matrix, then

$$U = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

Since V_3 was the zero vector, A_3 is renamed (interchange columns). A new V_3 is generated, namely,

$$V_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 0 \cdot V_1 - 0 \cdot V_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This "new" V_3 is not the zero vector. Hence, column operations terminate. The row operations are the following:

1. Interchange row 3 and row 4.
2. β_{31} times row 4 added to row 1.
3. β_{32} times row 4 added to row 2.

Thus,

$$U^{-1}AU = \begin{bmatrix} 5 & 8 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix}$$

The rank of A is 3. The eigenvector associated with the zero eigenvalue of A is the fourth column of U. Since

$$Q_r = \begin{bmatrix} 5 & 8 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is nonsingular, no further reduction need be applied. The eigenvalues of Q_r are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}(3 + \sqrt{17})$ and $\lambda_3 = \frac{1}{2}(3 - \sqrt{17})$. The respective eigenvectors are the following:

$$Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Z_2 = \frac{1}{2}\begin{bmatrix} -7-\sqrt{17} \\ 2 \\ 0 \end{bmatrix}, \quad Z_3 = \frac{1}{2}\begin{bmatrix} -7+\sqrt{17} \\ 2 \\ 0 \end{bmatrix}.$$

To obtain the eigenvectors for the nonzero eigenvalues of A, (6) is applied, where

$$B = [2 \quad 3 \quad 0] .$$

Thus,

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \frac{1}{4}\begin{bmatrix} -4 \\ -1-\sqrt{17} \\ -5-\sqrt{17} \\ 0 \end{bmatrix}, \quad X_3 = \frac{1}{4}\begin{bmatrix} -4 \\ -1+\sqrt{17} \\ -5+\sqrt{17} \\ 0 \end{bmatrix}$$

and the eigenvector for the zero eigenvalue is

$$X_4 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

the last column of U.

B. Example 2.

Let

$$A = \begin{bmatrix} 3 & -2 & -1 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is easy to see that A_1, A_2 and A_3 are independent vectors (also V_1, V_2 and V_3) and $A_4 = A_1$. Thus, the only column operation is to add -1 times column 1 to column 4. Therefore,

$$Q_r = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$B = [0 \quad 0 \quad 1],$$

where

$$U = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we apply the algorithm again to Q_r, without intermediate calculations, we have

$$Q_r' = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix},$$

$$B' = [0 \quad 1]$$

and U' is the 3×3 identity matrix, where the prime denotes the partitioning of Q_r. The two eigenvalues of Q_r' are $\lambda_1 = 2$ and $\lambda_2 = 1$, and the associated eigenvectors are

$$Z_1' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and

$$Z_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The respective eigenvectors of Q_r are .

$$Z_1 = U' \begin{bmatrix} Z_1' \\ \\ \lambda_1^{-1} B' Z_1' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

and

$$Z_2 = U' \begin{bmatrix} Z_2' \\ \\ \lambda_2^{-1} B' Z_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, the eigenvectors of the nonzero eigenvalues of A are

$$X_1 = U\begin{bmatrix} Z_1 \\ \lambda_1^{-1}BZ_1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

and

$$X_2 = U\begin{bmatrix} Z_2 \\ \lambda_2^{-1}BZ_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

The eigenvector associated with the zero eigenvalue is

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} ,$$

that is, the fourth column of U. Since the rank of A is three, this is the only eigenvector associated with the zero eigenvalue.

VI. CONCLUSION

In order to apply this algorithm to a computer code, an a priori decision must be made for a " zero vector", due to machine round-off. In the author's program, which generates the matrices U, A_r and B, a vector, V, is the zero vector if

$$(V,V) < \varepsilon,$$

where ε is some preassigned value.

An alternate application of the procedure could eliminate some computer code "bookkeeping". Since matrix multiplication is associative, a column operation, U_p, could be immediately followed by the row operation U_p^{-1}. This would, of course, increase the arithmetical operations, since the knowledge of the number of zero columns could not be totally incorporated.

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